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Decomposable critical tensors ☆

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Abstract

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of m and $\lambda' = \left(\lambda'_1, \ldots, \lambda'_{\lambda_1}\right)$ its conjugate partition. Denote also by λ the irreducible $\mathbb{C}$-character of S_m associated with λ. Let V be a finite dimensional vector space over $\mathbb{C}$.

The reach of an element of the symmetry class of tensors V_λ (symmetry class of tensors associated with λ) is defined. The concept of critical element is introduced, as an element whose reach has dimension equal to λ'_1. It is observed that, in $\wedge^m V$, the notions of critical element and decomposable element coincide. Known results for decomposable elements of $\wedge^m V$ are extended to critical elements of V_λ. In particular, for a basis of $\otimes^m V$ induced by a basis of V, generalized Plücker polynomials are constructed in a way that the set of their common roots contains the set of the families of components of decomposable critical elements of V_λ.

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1. Introduction

Let V be a n-dimensional vector space over $\mathbb{C}$ and let $(e_1, \ldots, e_n)$ be a basis of V. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ $(\lambda_s > 0)$ be a partition of m and $\chi = (\lambda_1, \ldots, \lambda_s, 1)$ the partition of $m + 1$ obtained from λ by adding one part equal to 1. The irreducible complex characters of S_m correspond canonically in a one to one way to the partitions of m. So, we identify λ with the corresponding irreducible complex character of S_m and χ with the corresponding irreducible complex character of S_{m+1}.

We denote by $\otimes^m V$ the mth tensor power of V. If $\sigma \in S_m$, then $\mathscr{P}(\sigma)$ is the unique linear operator on $\otimes^m V$ satisfying

$$\mathscr{P}(\sigma)(x_1 \otimes \cdots \otimes x_m) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}$$

for all $x_1, \ldots, x_m \in V$.

Let $G \leqslant S_m$ and ν a complex character of G. We define the *symmetrizer* associated with ν and G as the linear operator,

$$T(G, \nu) := \frac{\nu(\mathrm{id})}{|G|} \sum_{\sigma \in G} \nu(\sigma)\mathscr{P}(\sigma).$$

The symmetrizer associated with λ is the linear operator,

$$T_\lambda := T(S_m, \lambda) = \frac{\lambda(\mathrm{id})}{m!} \sum_{\sigma \in S_m} \lambda(\sigma)\mathscr{P}(\sigma).$$

The range of T_λ is called the *symmetry class of tensors* associated with λ and is denoted by V_λ. The image by T_λ of the decomposable tensor $x_1 \otimes \cdots \otimes x_m$, where $x_1, \ldots, x_m \in V$, is called the *decomposable symmetrized tensor* or *decomposable tensor* of V_λ and is denoted by

$$x_1 * \cdots * x_m := T_\lambda(x_1 \otimes \cdots \otimes x_m).$$

Let $z \in V_\lambda$. A family $(x_{ij})_{\substack{i=1,\ldots,k\\ j=1,\ldots,m}}$ that satisfies

$$\sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im} \in T_\lambda^{-1}(\{z\})$$

is called a *pre-image family* of z in V. Let $\mathscr{X} = (x_{ij})_{\substack{i=1,\ldots,k\\ j=1,\ldots,m}}$ be a pre-image family of $z \in V_\lambda$. We call the pair

$$(\mathscr{X}, z) = \left((x_{ij})_{\substack{i=1,\ldots,k\\ j=1,\ldots,m}}, z\right)$$

a *presentation* of z. By abuse of language *presentation* of z is the expression

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im},$$

where $(x_{ij})_{\substack{i=1,\dots,k\\ j=1,\dots,m}}$ is a pre-image family of z. The tensor

$$z^{\otimes}_{\mathscr{X}} = \sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im}$$

is a *root of the presentation* $(\mathscr{X}, z)$. The vectors x_{ij} are the *vectors of the presentation* and the dimension of the subspace of V

$$\langle x_{ij} : i = 1, \dots, k,\ j = 1, \dots, m\rangle$$

is called the *dimension of the presentation*.

If λ is the alternating character ε, V_λ is denoted by $\wedge^m V$, the well known *m-th Grassmann space*, or the *m-th exterior power* of V and the decomposable symmetrized tensors $T_\varepsilon(x_1 \otimes \cdots \otimes x_m)$ are the *decomposable tensors of Grassmann* denoted by

$$x_1 \wedge \cdots \wedge x_m.$$

It is well known that the tensors of the form $x_1 \wedge \cdots \wedge x_m$, with $x_1, \dots, x_m \in V$, are an algebraic variety of $\mathscr{A}^{\binom{n}{m}}(\mathbb{C})$. This algebraic variety is the affine cone of a projective variety whose defining polynomials are the quadratic Plücker polynomials.

We define the *reach* of a nonzero tensor of V_λ, as the smallest (by inclusion) subspace W of V such that $z \in W_\lambda$. We define also the *annihilator* of a nonzero tensor z of V_λ as the subspace of the reach of z whose elements v satisfy

$$T_\chi\left(\sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im} \otimes v\right) = 0$$

whenever

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is a presentation of z with vectors in the reach of z. The concepts of reach and annihilator of a tensor of V_λ have a crucial role in this approach. The elementary properties of these concepts and the relations with the critical elements of V_λ are established. When λ is the alternating character the annihilator of z is the subspace of V whose elements $v \in V$ satisfy

$$z \wedge v = 0.$$

We prove that a Grassmann tensor is critical if and only if it is Grassmann decomposable. This observation allow us to conclude that the first of the main theorems of this paper generalizes the well known result (see [6]):

Theorem 1.1. *Let z be a nonzero vector in $\wedge^m V$. Then z is decomposable (in $\wedge^m V$) if and only if there exists a linearly independent set of m vectors $u_1, \dots, u_m$ such that*

$$z \wedge u_i = 0, \quad i = 1, \dots, m.$$

Following the strategy presented by Marcus in [6] we construct a family of generalized Plücker polynomials for characterizing the set of critical decomposable tensors of V_λ.

2. Combinatorial tour

Let X be a finite set, we denote by $\Gamma_{m,X}$ the set of all mappings from $\{1,\dots,m\}$ into X. When $X=\{1,\dots,n\}$, we use the notation $\Gamma_{m,n}(\Gamma^0_{m,n})$ to the set of the mappings from $\{1,\dots,m\}$ into $\{1,\dots,n\}$ (respectively $\{1,\dots,m\}$ into $\{0,\dots,n\}$). We will call the *multiplicity partition* of $\alpha\in\Gamma_{m,X}$ the partition of m obtained by rearranging in decreasing order the components of the family of nonnegative integers $(|\alpha^{-1}(x)|)_{x\in X}$. We denote the multiplicity partition of α by $M(\alpha)$.

Let $\omega\in\Gamma_{m,n}$ we denote by ω_i the element of $\Gamma_{m-1,n}$

$$\omega_i := (\omega(1),\dots,\omega(i-1),\omega(i+1),\dots,\omega(m)),\quad i=1,\dots,m.$$

If $\nu\in\Gamma_{m-1,n}$, $t\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$ we will denote by $\nu\overset{t}{\hookleftarrow}j$ the element of $\Gamma_{m,n}$ defined by

$$\nu\overset{t}{\hookleftarrow}j := (\nu(1),\dots,\nu(t-1),j,\nu(t),\dots,\nu(m-1)),\quad \text{if } t=1,\dots,m-1,$$

and

$$\nu\overset{m}{\hookleftarrow}j := (\nu(1),\dots,\nu(m-1),j).$$

If $\alpha\in\Gamma_{m,n}$ we denote by $\hat\alpha$ the element of $\Gamma^0_{m+1,n}$,

$$\hat\alpha=(\alpha(1),\dots,\alpha(m),0).$$

The subset of the increasing functions of $\Gamma_{m,n}$ is denoted by $G_{m,n}$.

We define an action

$$(\sigma,\alpha)\to\alpha\sigma^{-1}$$

of S_m (respectively S_{m+1}) on $\Gamma_{m,n}$ (respectively on $\Gamma^0_{m+1,n}$). If $\alpha\in\Gamma_{m,n}$ the orbit of α is denoted by $\mathcal{O}_\alpha$. If α and β belongs to the same orbit we will write $\alpha\equiv\beta(\operatorname{mod} S_m)$. Observe that $G_{m,n}$ is the system of distinct representatives of the orbits of this action, choosing in each orbit $\mathcal{O}_\alpha$ the smallest element by the lexicographic order. We denote by H_α the stabilizer of α.

Lemma 2.1. *Let α and β be elements of $\Gamma_{m,n}$. Then $\alpha\equiv\beta(\operatorname{mod} S_m)$ if and only if*

$$|\alpha^{-1}(i)|=|\beta^{-1}(i)|,\quad i=1,\dots,n.$$

Proposition 2.1. *Let α and β be elements of $\Gamma_{m,n}$. Then $\alpha\equiv\beta(\operatorname{mod} S_m)$ if and only if $\hat\alpha\equiv\hat\beta(\operatorname{mod} S_{m+1})$.*

Let $(e_1,\dots,e_n)$ be a basis of V and $\alpha\in\Gamma_{m,n}$. We denote by $e^{\otimes}_\alpha$ the element of $\otimes^m V$

$$e^{\otimes}_\alpha := e_{\alpha(1)}\otimes\cdots\otimes e_{\alpha(m)}.$$

In the same way e^*_α is the element of V_λ

$$e^*_\alpha := e_{\alpha(1)}*\cdots*e_{\alpha(m)}.$$

Denote by Ω_λ (or just by Ω) the subset of $\Gamma_{m,n}$,

$$\Omega := \{\alpha \in \Gamma_{m,n} : e^*_\alpha \neq 0\}.$$

By the definitions it is easy to conclude that

$$V_\lambda \subseteq \langle e^\otimes_\alpha : \alpha \in \Omega\rangle. \tag{1}$$

So, if $z = \sum_{\alpha\in\Gamma_{m,n}} c_\alpha e^\otimes_\alpha \in V_\lambda$, we define the *support* of z as follows:

$$\operatorname{supp}(z) := \{\alpha \in \Gamma_{m,n} : c_\alpha \neq 0\},$$

and we observe that $\operatorname{supp}(z) \subseteq \Omega$.

Let m be a positive integer and $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of m. We identify λ with an m-tuple of nonnegative integers by adding, if necessary, a list of zeros, i.e.

$$\lambda = (\lambda_1, \ldots, \lambda_t) \equiv (\lambda_1, \ldots, \lambda_t, 0, \ldots, 0).$$

If λ is a partition of m, then $\lambda' = \left(\lambda'_1, \ldots, \lambda'_{\lambda_1}\right)$ defined by

$$\lambda'_k = |\{j \in \{1, \ldots, t\} : \lambda_j \geqslant k\}|, \quad k = 1, \ldots, \lambda_1$$

is also a partition of m called the *conjugate partition* of λ.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\nu = (\nu_1, \ldots, \nu_m)$ be partitions of m. We say that λ majorizes ν, and denote $\lambda \succeq \nu$, if

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \nu_i, \quad k = 1, \ldots, m.$$

3. Auxiliary results

Let W be a subspace of V and $(e_1, \ldots, e_n)$ be a basis of W. Let $e_0 \notin W$ and denote by U the subspace of V, $U = W + \langle e_0\rangle$. Then $(e_0, \ldots, e_n)$ is a basis of U and

$$(e^\otimes_\beta : \beta \in \Gamma^0_{m+1,n})$$

is a basis of $\otimes^{m+1} U$. Therefore

$$\begin{aligned}
\otimes^{m+1} U &= \langle e^\otimes_\beta : \beta \in \Gamma^0_{m+1,n}, |\beta^{-1}(\{0\})| = 1, \beta(m+1) = 0\rangle \\
&\qquad \oplus \langle e^\otimes_\beta : \beta \in \Gamma^0_{m+1,n}, |\beta^{-1}(\{0\})| = 1, \beta(m+1) \neq 0\rangle \\
&\qquad \oplus \langle e^\otimes_\beta : \beta \in \Gamma^0_{m+1,n}, |\beta^{-1}(\{0\})| \neq 1\rangle \\
&= \langle e^\otimes_{\hat\alpha} : \alpha \in \Gamma_{m,n}\rangle \oplus \langle e^\otimes_\beta : \beta \in \Gamma^0_{m+1,n}, \\
&\qquad |\beta^{-1}(\{0\})| = 1, \beta(m+1) \neq 0\rangle \\
&\qquad \oplus \langle e^\otimes_\beta : \beta \in \Gamma^0_{m+1,n}, |\beta^{-1}(\{0\})| \neq 1\rangle.
\end{aligned} \tag{2}$$

Let $(x_1, \ldots, x_m)$ be a family of nonzero vectors of V and $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of m. A μ-*coloring* of $(x_1, \ldots, x_m)$ or *coloring of shape* μ, is a decomposition of $(x_1, \ldots, x_m)$ in linearly independent subfamilies,

$$(x_1,\ldots,x_m)=(x_i)_{i\in\Delta_1}\dot{\cup}\cdots\dot{\cup}(x_i)_{i\in\Delta_k},$$

where $(\Delta_1,\ldots,\Delta_k)$ is a set partition of $\{1,\ldots,m\}$ and $|\Delta_i|=\mu_i,\ i=1,\ldots,k$. We say that the family $(x_1,\ldots,x_m)$ is *μ-colorable* if there exists a coloring of $(x_1,\ldots,x_m)$ of shape μ.

In [1] it was proved that in the majorization order, the set of the shapes of the colorings of $(x_1,\ldots,x_m)$ has a maximum. This maximum partition is the *rank partition* of $(x_1,\ldots,x_m)$ and is denoted by

$$\rho(x_1,\ldots,x_m).$$

In [3] Gamas proved the following result that we present here with the formulation referred to [1]:

Proposition 3.2. *Let λ be an irreducible character of S_m. Let $(x_1,\ldots,x_m)$ be a family of nonzero vectors of V. Then $T_\lambda(x_1\otimes\cdots\otimes x_m)\neq 0$ if and only if*

$$\rho(x_1,\ldots,x_m)\succeq\lambda'.$$

Remark

1. By the Proposition 3.2, if z is a nonzero decomposable tensor of V_λ, then the dimension of the presentations of z is greater or equal to λ'_1.
2. The Proposition 3.2 is a generalization of the following result previously established by Merris [7].

Proposition 3.3. *Let λ be an irreducible character of S_m. Let $(e_1,\ldots,e_n)$ be a basis of V. If $\alpha\in\Gamma_{m,n}$ then $T_\lambda(e^{\otimes}_\alpha)\neq 0$ if and only if $\lambda\succeq M(\alpha)$.*

The next proposition is another formulation of the Gamas Theorem presented for the first time in [1].

Proposition 3.4. *Let λ be an irreducible character of S_m. Let $(x_1,\ldots,x_m)$ be a family of nonzero vectors of V. Then $T_\lambda(x_1\otimes\cdots\otimes x_m)\neq 0$ if and only if the family $(x_1,\ldots,x_m)$ is λ'-colorable.*

The relation between the principal result of this article and the classical results on Grassmann spaces is established in Corollary 1 to Theorem 4.3. The corollary depends on the following theorem:

Theorem 3.2. *Let $(x_1,\ldots,x_m),(y_1,\ldots,y_m)$ be families of linearly independent vectors of V. Then*

$$\langle x_1\wedge\cdots\wedge x_m\rangle=\langle y_1\wedge\cdots\wedge y_m\rangle$$

if and only if

$$\langle x_1,\ldots,x_m\rangle=\langle y_1,\ldots,y_m\rangle.$$

4. Pre-image families and reach of a tensor of V_λ

Let

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

be a presentation of z. If, for all subset $\emptyset \neq L \subseteq \{1, \ldots, k\}$, we have

$$\sum_{l \in L} x_{l1} * \cdots * x_{lm} \neq 0 \tag{3}$$

we say that $z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}\left((x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}\right)$ is a *simple presentation* of z (a *simple pre-image family* of z).

From now on we assume that all the presentations (pre-image families) considered are simple.

Definition 4.1. Let $0 \neq z \in V_\lambda$,

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is a *critical presentation* of z and the family $(x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ is a *critical pre-image family* of z if

$$\dim\langle x_{i1}, \ldots, x_{im}\rangle = \lambda'_1, \quad i = 1, \ldots, k.$$

Definition 4.2. Let $0 \neq z \in V_\lambda$. We say that z is *weakly decomposable* if exists a presentation of z

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

such that

$$\langle x_{i1}, \ldots, x_{im}\rangle = \langle x_{j1}, \ldots, x_{jm}\rangle, \quad i, j \in \{1, \ldots, k\}.$$

This presentation of z is called *weakly decomposable* and the corresponding pre-image family is also called *weakly decomposable*.

Definition 4.3. A nonzero vector of V_λ has *rank* k if it is a sum of k and not less than k decomposable symmetrized tensors of V_λ. If $z \in V_\lambda$ has rank k then the expression

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is called a *rank presentation* of z and the family $(x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ will be called a *rank pre-image family* of z.

In [4] Lim proved the following result:

Lemma 4.2. *Let z be a nonzero tensor of V_λ. If*

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is a rank presentation of z and

$$z = \sum_{j=1}^{q} y_{j1} * \cdots * y_{jm}$$

is another presentation of z, then

$$\sum_{i=1}^{k} \langle x_{id} : d = 1, \ldots, m\rangle \subseteq \sum_{j=1}^{q} \langle y_{jd} : d = 1, \ldots, m\rangle.$$

Proposition 4.5. *Let $z \in V_\lambda$. Let $\mathscr{A}_z = \{U \text{ subspace of } V : z \in U_\lambda\}$. Then,*

$$z \in \left(\bigcap_{U \in \mathscr{A}_z} U\right)_\lambda .$$

Proof. Let

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

be a rank presentation of z. Let $U \in \mathscr{A}_z$. Then $z \in U_\lambda$ and there exists $u_{it} \in U$ with $i = 1, \ldots, s$, $t = 1, \ldots, m$, such that

$$z = \sum_{i=1}^{s} u_{i1} * \cdots * u_{im}.$$

Using Lemma 4.2 we obtain

$$\sum_{i=1}^{k} \langle x_{id} : d = 1, \ldots, m\rangle \subseteq \sum_{i=1}^{s} \langle u_{id} : d = 1, \ldots, m\rangle \subseteq U.$$

From this we conclude that

$$\sum_{i=1}^{k} \langle x_{id} : d = 1, \ldots, m\rangle \subseteq U,$$

for all $U \in \mathscr{A}_z$. Then,

$$\sum_{i=1}^{k} \langle x_{id} : d = 1, \ldots, m\rangle \subseteq \bigcap_{U \in \mathscr{A}_z} U.$$

Hence,

$$z \in \left(\bigcap_{U \in \mathscr{A}_z} U \right)_\lambda. \qquad \square$$

Definition 4.4. The *reach* of z, denoted by $W(z)$, is the intersection of the subspaces W of V such that $z \in W_\lambda$.

By the definition, $W(z)$ is the smallest subspace, by inclusion, that contains a pre-image family of z.

Definition 4.5. Let $0 \neq z \in V_\lambda$. We say that z is *critical* if $\dim W(z) = \lambda'_1$.

Proposition 4.6. *Let $0 \neq z$ be a critical tensor of V_λ. Then all the presentations of z with vectors in $W(z)$ are simultaneously critical and weakly decomposable.*

Proof. Let

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

be a presentation of z with vectors in $W(z)$. Since z is critical, for all $i \in \{1, \ldots, k\}$, we have

$$\dim\langle x_{i1}, \ldots, x_{im}\rangle \leqslant \dim W(z) = \lambda'_1.$$

By Proposition 3.2 (we assume $z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$ simple) we conclude that

$$\dim\langle x_{i1}, \ldots, x_{im}\rangle \geqslant \lambda'_1, \quad i \in \{1, \ldots, k\}.$$

So, $\dim\langle x_{i1}, \ldots, x_{im}\rangle = \dim W(z) = \lambda'_1$ and then $\langle x_{i1}, \ldots, x_{im}\rangle = W(z)$ for all $i = 1, \ldots, k$. $\square$

Theorem 4.3. *Let $0 \neq z \in V_\lambda$ and let*

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

be a rank presentation of z. Then

$$W(z) = \langle x_{ij} : i = 1, \ldots, k,\ j = 1, \ldots, m\rangle.$$

Proof. By definition of reach we conclude that

$$W(z) \subseteq \langle x_{ij} : i = 1, \ldots, k,\ j = 1, \ldots, m\rangle.$$

Since $z \in W(z)_\lambda$, there exists $y_{ij} \in W(z)$, $i = 1, \ldots, l$, $j = 1, \ldots, m$, such that $z = \sum_{i=1}^{l} y_{i1} * \cdots * y_{im}$ is a presentation of z. Then, by Lemma 4.2, we have

$$W(z) \subseteq \sum_{i=1}^{k} \langle x_{ij} : j = 1, \ldots, m \rangle \subseteq \sum_{i=1}^{l} \langle y_{ij} : j = 1, \ldots, m \rangle \subseteq W(z). \qquad \square$$

Corollary 1. *If $V_\lambda = \wedge^m V$ then a nonzero $z \in \wedge^m V$ is critical if and only if it is decomposable.*

Proof. We observe first that the partition corresponding to ε is (1^m). Then, all nonzero decomposable tensors $x_1 \wedge \cdots \wedge x_m$ are critical, since they satisfy

$$\dim\langle x_1, \ldots, x_m \rangle = \varepsilon'_1 = m.$$

By the previous theorem, Proposition 4.6 and Theorem 3.2 it is easy to conclude that if z is critical, then z is a decomposable element of $\wedge^m V$. $\square$

Corollary 2. *If $V_\lambda = \vee^m V$ then a nonzero $z \in \vee^m V$ is critical if and only if there exists a nonzero $y \in V$ such that $z = y \vee \cdots \vee y$.*

Proof. We observe first that the partition corresponding to the principal character of S_m is (m). By the previous theorem, if

$$z = \sum_{i=1}^{k} x_{i1} \vee \cdots \vee x_{im}$$

is a rank presentation of z, then $W(z) = \langle x_{ij} : i = 1, \ldots, k,\ j = 1, \ldots, m \rangle$.

If z is critical, we have $\dim W(z) = 1$, so there exist $c_{ij}, i = 1, \ldots, k,\ j = 1, \ldots, m$ and $x \in V$ such that $x_{ij} = c_{ij} x$. Then, consider $c_i = \prod_{j=1}^{m} c_{ij}$ and $c = \sum_{i=1}^{k} c_i$. Hence, for a in $\mathbb{C}$ such that $a^m = c$,

$$\begin{aligned} z &= \left(\sum_{i=1}^{k} c_i \right) x \vee \cdots \vee x \\ &= c(x \vee \cdots \vee x) \\ &= (ax) \vee \cdots \vee (ax). \end{aligned}$$

Conversely, if $0 \neq z = y \vee \cdots \vee y$ then $y \neq 0$. By the previous theorem, we have $W(z) = \langle y \rangle$, so $\dim W(z) = 1$ and z is critical. $\square$

Proposition 4.7. *Let $0 \neq z \in V_\lambda$ and let*

$$z = \sum_{i=1}^{l} u_{i1} * \cdots * u_{im} \tag{4}$$

be a weakly decomposable presentation of z. Then there exists a weakly decomposable presentation of z with vectors in $W(z)$ and dimension less or equal to the dimension of the presentation (4).

Proof. Let $(x_{ij})_{\substack{i=1,\dots,k\\ j=1,\dots,m}}$ be a rank pre-image family of z. Let $(u_{ij})_{\substack{i=1,\dots,l\\ j=1,\dots,m}}$ be the weakly decomposable pre-image family of z. Then

$$\sum_{i=1}^{k} x_{i1} * \cdots * x_{im} = z = \sum_{i=1}^{l} u_{i1} * \cdots * u_{im}. \tag{5}$$

Let P be a projection of V over $W(z)$. Then, by Theorem 4.3, $P(x_{ij}) = x_{ij}$ for all $i = 1, \dots, k$ and $j = 1, \dots, m$. The images by $\otimes^m P = P \otimes \cdots \otimes P$ in the both sides of the equality (5) are

$$\begin{aligned}(\otimes^m P)(z) &= (\otimes^m P)\left(\sum_{i=1}^{k} x_{i1} * \cdots * x_{im}\right)\\ &= \sum_{i=1}^{k} P(x_{i1}) * \cdots * P(x_{im})\\ &= \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}\\ &= z\\ &= \sum_{i=1}^{l} P(u_{i1}) * \cdots * P(u_{im}).\end{aligned}$$

Suppose, without loss of generality, that $s \leqslant l$ is a positive integer and

$$z = \sum_{i=1}^{s} P(u_{i1}) * \cdots * P(u_{im}) \tag{6}$$

is simple. But,

$$P(\langle u_{i1}, \dots, u_{im}\rangle) = \langle P(u_{i1}), \dots, P(u_{im})\rangle$$

so, we conclude from $(u_{ij})_{\substack{i=1,\dots,l\\ j=1,\dots,m}}$ being weakly decomposable, that $(P(u_{ij}))_{\substack{i=1,\dots,s\\ j=1,\dots,m}}$ is a pre-image family of z weakly decomposable with elements in $W(z)$ and the dimension of the presentation (6) is less or equal to the dimension of the presentation (4). □

Corollary 3. *Every nonzero weakly decomposable tensor $z \in V_\lambda$ has a presentation*

$$z = \sum_{i=1}^{l} u_{i1} * \cdots * u_{im}$$

such that $W(z) = W(u_{i1} * \cdots * u_{im})$, for all $i = 1, \dots, l$.

Proof. By Proposition 4.7, there exists a weakly decomposable presentation of z with vectors in $W(z)$,

$$z = \sum_{i=1}^{s} u_{i1} * \cdots * u_{im}.$$

Suppose, without loss of generality, that this presentation is simple. So, by Theorem 4.3, $W(u_{i1} * \cdots * u_{im}) = \langle u_{i1}, \ldots, u_{im} \rangle$, for all $i = 1, \ldots, s$.

As $u_{ij} \in W(z)$, then, for all $i = 1, \ldots, s$,

$$W(u_{i1} * \cdots * u_{im}) = \langle u_{i1}, \ldots, u_{im} \rangle \subseteq W(z).$$

For the other inclusion, as we have a weakly decomposable presentation, by Lemma 4.2, we obtain

$$\begin{aligned} W(z) &\subseteq \langle u_{ij} : i = 1, \ldots, s, \ j = 1, \ldots, m \rangle \\ &= \langle u_{i1}, \ldots, u_{im} \rangle = W(u_{i1} * \cdots * u_{im}) \end{aligned}$$

for all $i = 1, \ldots, s$. $\square$

Proposition 4.8. *Let $0 \neq z \in V_\lambda$. The tensor z is critical if and only if it admits a presentation simultaneously critical and weakly decomposable. Moreover, if z is critical, a presentation of z is critical and weakly decomposable if and only if the vectors are in $W(z)$.*

Proof. Suppose that

$$z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$$

is a critical and a weakly decomposable presentation of z. If we consider $z_i = x_{i1} * \cdots * x_{im}$, $i = 1, \ldots, k$, then all the $W(z_i) = \langle x_{i1}, \ldots, x_{im} \rangle$ are identical subspaces with dimension equal to λ_1'. So, by Lemma 4.2,

$$W(z) \subseteq \sum_{i=1}^{k} \langle x_{ij} : j = 1, \ldots, m \rangle = \langle x_{11}, \ldots, x_{1m} \rangle$$

and then $\dim W(z) \leqslant \lambda_1'$. But, for any rank presentation of

$$z = \sum_{i=1}^{s} y_{i1} * \cdots * y_{im},$$

we see from Theorem 4.3 that $W(y_{i1} * \cdots * y_{im}) \subseteq W(z)$. Since $\dim W(y_{i1} * \cdots * y_{im}) \geqslant \lambda_1'$, it follows that $\dim W(z) \geqslant \lambda_1'$. This shows that $\dim W(z) = \lambda_1'$. Hence z is critical.

Conversely, if z is critical, by Proposition 4.6, the presentations of z with vectors in $W(z)$ are critical and weakly decomposable.

Finally, we know by Proposition 4.6, that if z is critical all the presentations of z with vectors in $W(z)$ are critical and weakly decomposable. Conversely, if

$$z = \sum_{i=1}^{k} y_{i1} * \cdots * y_{im}$$

is a critical and weakly decomposable presentation, we have

$$W(z) \subseteq \sum_{i=1}^{k} \langle y_{ij} : j = 1, \ldots, m \rangle = \langle y_{s1}, \ldots, y_{sm} \rangle$$

for all $s \in \{1, \ldots, k\}$. Then, by an argument of dimension, we have

$$W(z) = \langle y_{s1}, \ldots, y_{sm} \rangle, \quad s = 1, \ldots, k. \qquad \square$$

Definition 4.6. Let $\mathscr{U} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ be a pre-image family of a nonzero tensor z of V_λ. Then the *annihilator* of $\mathscr{U}$ (or the presentation $z = \sum_{i=1}^{k} x_{i1} * \cdots * x_{im}$) is defined as follows:

$$\mathrm{Ann}^{\mathscr{U}}(z) := \{v \in V : T_\chi(z_{\mathscr{U}}^{\otimes} \otimes v) = 0\}.$$

We are now prepared to prove the following theorem:

Theorem 4.4. *Let V be a vector space over $\mathbb{C}$. Let z be a nonzero and critical element of V_λ. Then all the pre-image families of z,*

$$\mathscr{U} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$$

with elements in $W(z)$, satisfy

$$W(z) = \mathrm{Ann}^{\mathscr{U}}(z).$$

Proof. Observe first that $W(z) = \langle x_{i1}, \ldots, x_{im} \rangle$, $i = 1, \ldots, k$. So, if $v \in W(z)$, we have

$$(\rho(x_{i1}, \ldots, x_{im}, v))_1 = (\rho(x_{i1}, \ldots, x_{im}))_1 = \lambda'_1, \quad i = 1, \ldots, k.$$

Then,

$$(\rho(x_{i1}, \ldots, x_{im}, v))_1 < \chi'_1 = \lambda'_1 + 1, \quad i = 1, \ldots, k.$$

So,

$$\rho(x_{i1}, \ldots, x_{im}, v) \not\trianglerighteq \chi'.$$

Then, by Proposition 3.2, we have

$$T_\chi(x_{i1} \otimes \cdots \otimes x_{im} \otimes v) = 0, \quad i = 1, \ldots, k.$$

Therefore,

$$T_\chi\left(\sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im} \otimes v\right) = 0.$$

Conversely, we will show that if $v \notin W(z)$ then

$$T_\chi(z^{\otimes}_{\mathscr{U}} \otimes v) \neq 0.$$

In order to prove this result we start by introducing terminology, notation and some results about the symmetric group. We will denote by S'_m the subgroup of S_{m+1}

$$S'_m = \{\sigma \in S_{m+1} : \sigma(m+1) = m+1\}.$$

Consider in S_{m+1} the permutations $\tau_0 = \mathrm{id}$, $\tau_i = (m+1\ i)$ for $i = 1, \dots, m$. Then,

$$S_{m+1} = S'_m \dot{\cup} S'_m\tau_1 \dot{\cup} \cdots \dot{\cup} S'_m\tau_m$$

is a right coset decomposition of S'_m in S_{m+1}. Then, we have

$$\begin{aligned} T_\chi &= \frac{\chi(\mathrm{id})}{(m+1)!}\sum_{\sigma\in S_{m+1}} \chi(\sigma)\mathscr{P}(\sigma) \\ &= \frac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=0}^{m}\sum_{\sigma\in S'_m} \chi(\sigma\tau_i)\mathscr{P}(\sigma\tau_i) \\ &= \frac{\chi(\mathrm{id})}{(m+1)!}\left[\sum_{\sigma\in S'_m}\chi(\sigma)\mathscr{P}(\sigma) + \sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathscr{P}(\sigma)\mathscr{P}(\tau_i)\right] \\ &= \frac{1}{m+1}T_{\chi_{|S'_m}} + \frac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathscr{P}(\sigma)\mathscr{P}(\tau_i). \end{aligned} \tag{7}$$

Let $(e_1, \dots, e_{\lambda'_1})$ be a basis of $W(z)$. Then $(e_0 = v, e_1, \dots, e_{\lambda'_1})$ is a linearly independent family.

As $z^{\otimes}_{\mathscr{U}}$ is the root of the presentation $(\mathscr{U}, z)$,

$$z^{\otimes}_{\mathscr{U}} = \sum_{i=1}^{k} x_{i1} \otimes \cdots \otimes x_{im},$$

then,

$$\begin{aligned} T_\chi(z^{\otimes}_{\mathscr{U}} \otimes v) = {} & \underbrace{\frac{1}{m+1}T_{\chi_{|S'_m}}(z^{\otimes}_{\mathscr{U}} \otimes v)}_{\mathscr{A}} \\ & + \underbrace{\frac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathscr{P}(\sigma)\mathscr{P}(\tau_i)(z^{\otimes}_{\mathscr{U}} \otimes v)}_{\mathscr{B}}. \end{aligned}$$

Our purpose is to prove that $T_\chi(z^{\otimes}_{\mathscr{U}} \otimes v)$ is not equal to zero. We compute separately parts $\mathscr{A}$ and $\mathscr{B}$.

Bearing in mind that $\left(e_1, \ldots, e_{\lambda'_1}\right)$ is a basis of $W(z)$, we have

$$z^{\otimes}_{\mathscr{U}} = \sum_{\alpha \in \Gamma_{m,\lambda'_1}} c_\alpha e^{\otimes}_{\alpha}. \tag{8}$$

Part $\mathscr{B}$.

$$\begin{aligned}
&\frac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathscr{P}(\sigma)\mathscr{P}(\tau_i)(z^{\otimes}_{\mathscr{U}}\otimes v)\\
&\quad=\frac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathscr{P}(\sigma)\mathscr{P}(\tau_i)\left(\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e^{\otimes}_{\alpha}\otimes e_0\right)\\
&\quad=\frac{\chi(\mathrm{id})}{(m+1)!}\sum_{i=1}^{m}\sum_{\sigma\in S'_m}\chi(\sigma\tau_i)\mathscr{P}(\sigma)\left(\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e^{\otimes}_{\alpha_i \overset{i}{\hookleftarrow} 0}\otimes e_{\alpha(i)}\right)
\end{aligned}$$

and as $\alpha(i) \neq 0$, $i = 1, \ldots, m$,

$$\in \left\langle e^{\otimes}_{\beta} : \beta \in \Gamma^{0}_{m+1,\lambda'_1}, |\beta^{-1}(0)| = 1, \beta(m+1) \neq 0\right\rangle.$$

Part $\mathscr{A}$. According to (8) we have

$$\begin{aligned}
T_{\chi_{|S'_m}}(z^{\otimes}_{\mathscr{U}}\otimes v) &= T_{\chi_{|S'_m}}\left(\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e^{\otimes}_{\alpha}\otimes e_0\right)\\
&= T_{\chi_{|S'_m}}\left(\sum_{\alpha\in\Gamma_{m,\lambda'_1}} c_\alpha e^{\otimes}_{\hat{\alpha}}\right).
\end{aligned}$$

For $\sigma \in S'_m$, we have

$$\begin{aligned}
\mathscr{P}(\sigma)(e^{\otimes}_{\hat{\alpha}}) &= \mathscr{P}(\sigma)(e^{\otimes}_{\alpha}\otimes e_0)\\
&= e^{\otimes}_{\alpha\sigma}\otimes e_0\\
&= e^{\otimes}_{\widehat{\alpha\sigma}}
\end{aligned}$$

and we conclude that

$$T_{\chi_{|S'_m}}(z^{\otimes}_{\mathscr{U}}\otimes v) \in \langle e^{\otimes}_{\hat{\alpha}} : \alpha \in \Gamma_{m+1,\lambda'_1}\rangle.$$

So, according to (2), if we show that part $\mathscr{A}$ is not equal to zero, we conclude that $T_\chi(z^{\otimes}_{\mathscr{U}}\otimes v) \neq 0$.

By the "Branching Theorem", λ is a constituent of $\chi_{|S'_m}$. Then, there exist irreducible characters of S_m, $\lambda = \lambda^{(1)}, \ldots, \lambda^{(l)}$, such that

$$\chi_{|S'_m} = \lambda + \lambda^{(2)} + \cdots + \lambda^{(l)}. \tag{9}$$

So, by (9),

$$\begin{aligned}
T(S'_m, \chi_{|S'_m})&(v_1 \otimes \cdots \otimes v_{m+1}) \\
&= \frac{\chi(\mathrm{id})}{m!} \sum_{\sigma \in S'_m} \chi(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m+1)} \\
&= \frac{\chi(\mathrm{id})}{m!} \sum_{\sigma \in S'_m} \chi(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)} \otimes v_{m+1} \\
&= \frac{\chi(\mathrm{id})}{m!} \sum_{\sigma \in S_m} ([\lambda + \cdots + \lambda^{(l)}](\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}) \otimes v_{m+1} \\
&= \sum_{i=1}^{l} \frac{\chi(\mathrm{id})}{\lambda^{(i)}(\mathrm{id})} \left(\frac{\lambda^{(i)}(\mathrm{id})}{m!} \sum_{\sigma \in S_m} \lambda^{(i)}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)} \right) \otimes v_{m+1} \\
&= \sum_{i=1}^{l} \frac{\chi(\mathrm{id})}{\lambda^{(i)}(\mathrm{id})} T_{\lambda^{(i)}}(v_1 \otimes \cdots \otimes v_m) \otimes v_{m+1}.
\end{aligned} \tag{10}$$

But, by (10) and because the projections $T_\lambda, T_{\lambda^{(2)}}, \ldots, T_{\lambda^{(l)}}$ are pairwise orthogonal projections,

$$\begin{aligned}
T(S'_m, \chi_{|S'_m})(\otimes^{m+1} V) \subseteq (T_\lambda(\otimes^m V) \otimes V) &\oplus (T_{\lambda^{(2)}}(\otimes^m V) \otimes V) \\
&\oplus \cdots \oplus (T_{\lambda^{(l)}}(\otimes^m V) \otimes V).
\end{aligned}$$

But the component of $T_{\chi_{|S'_m}}(z^{\otimes}_{\mathscr{U}} \otimes v)$ in $T_\lambda(\otimes^m V) \otimes V$ is

$$T_\lambda(z^{\otimes}_{\mathscr{U}}) \otimes v = z \otimes v,$$

which is not equal to zero because z and v are nonzero. $\square$

Remark. Using the arguments of the second part of the proof of the last theorem we can conclude that if $\mathscr{U} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ is a pre-image family of z with elements in $W(z)$ then

$$\mathrm{Ann}^{\mathscr{U}}(z) \subseteq W(z). \tag{11}$$

Definition 4.7. Let z be a nonzero element of V_λ. Then the *annihilator* of z, denoted by $\mathrm{Ann}(z)$, is the set of the elements $v \in V$ such that

$$T_\chi(z^{\otimes}_{\mathscr{U}} \otimes v) = 0$$

for all pre-image families $\mathscr{U} = (x_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ of z with elements in $W(z)$.

Remark. Let $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ a decomposable and critical tensor of V_λ. If $\mathscr{U} = (u_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ is a pre-image family of z with elements in $W(z)$, by Theorem 4.3 and the previous theorem, we have

$$\mathrm{Ann}^{\mathscr{U}}(z) = \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) = W(z),$$

and then $\mathrm{Ann}(z) = W(z)$.

5. Decomposable tensors

Proposition 5.9. *Let $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m) \in V_\lambda$. If*

$$\left\{y_{11}, \ldots, y_{1\lambda_1'}\right\} \dot{\cup} \cdots \dot{\cup} \left\{y_{\lambda_1 1}, \ldots, y_{\lambda_1 \lambda_{\lambda_1}'}\right\}$$

is a λ'-coloring of $(x_1, \ldots, x_m)$, then

$$\mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \subseteq \langle y_{11}, \ldots, y_{1\lambda_1'}\rangle.$$

Proof. Let

$$\left\{y_{11}, \ldots, y_{1\lambda_1'}\right\} \dot{\cup} \cdots \dot{\cup} \left\{y_{\lambda_1 1}, \ldots, y_{\lambda_1 \lambda_{\lambda_1}'}\right\}$$

be a λ'-coloring of $(x_1, \ldots, x_m)$. If $x \notin \langle y_{11}, \ldots, y_{1\lambda_1'}\rangle$ then $\left(x, y_{11}, \ldots, y_{1\lambda_1'}\right)$ is linearly independent and so

$$\left\{x, y_{11}, \ldots, y_{1\lambda_1'}\right\} \dot{\cup} \left\{y_{21}, \ldots, y_{2\lambda_2'}\right\} \dot{\cup} \cdots \dot{\cup} \left\{y_{\lambda_1 1}, \ldots, y_{\lambda_1 \lambda_{\lambda_1}'}\right\}$$

is a χ'-coloring of $(x_1, \ldots, x_m, x)$. So, by Proposition 3.4, we conclude that

$$T_\chi(x_1 \otimes \cdots \otimes x_m \otimes x) \neq 0$$

and then $x \notin \mathrm{Ann}^{(x_1,\ldots,x_m)}(z)$. So,

$$\mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \subseteq \langle y_{11}, \ldots, y_{1\lambda_1'}\rangle. \qquad \square$$

Corollary 1. *If $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ is a decomposable tensor of V_λ we have*

$$\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \leqslant \lambda_1'.$$

Theorem 5.5. *Let $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ be a decomposable tensor of V_λ, then z is critical if and only if*

$$\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) = \lambda_1'.$$

Proof. If z is critical and decomposable, by Theorem 4.4, we have that

$$W(z) = \mathrm{Ann}^{(x_1,\ldots,x_m)}(z).$$

So, $\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) = \lambda_1'$.

For the converse condition we need the following:

Fact. *If* $(x_1, \ldots, x_m)$ *is* λ'*-colorable and* $\dim\langle x_1, \ldots, x_m\rangle > \lambda'_1$ *then there exist two* λ'*-colorings*

$$\left\{x_{11}, \ldots, x_{1\lambda'_1}\right\}\dot{\cup}\left\{x_{21}, \ldots, x_{2\lambda'_2}\right\}\dot{\cup}\cdots\dot{\cup}\left\{x_{\lambda_1 1}, \ldots, x_{\lambda_1\lambda'_{\lambda_1}}\right\}$$

and

$$\left\{y_{11}, \ldots, y_{1\lambda'_1}\right\}\dot{\cup}\left\{y_{21}, \ldots, y_{2\lambda'_2}\right\}\dot{\cup}\cdots\dot{\cup}\left\{y_{\lambda_1 1}, \ldots, y_{\lambda_1\lambda'_{\lambda_1}}\right\}$$

such that

$$\langle x_{11}, \ldots, x_{1\lambda'_1}\rangle \neq \langle y_{11}, \ldots, y_{1\lambda'_1}\rangle.$$

Proof. Let

$$\left\{x_{11}, \ldots, x_{1\lambda'_1}\right\}\dot{\cup}\left\{x_{21}, \ldots, x_{2\lambda'_2}\right\}\dot{\cup}\cdots\dot{\cup}\left\{x_{\lambda_1 1}, \ldots, x_{\lambda_1\lambda'_{\lambda_1}}\right\}$$

be a λ'-coloring of $(x_1, \ldots, x_m)$. By hypothesis, $\dim\langle x_1, \ldots, x_m\rangle > \lambda'_1$, so, there exist $i \in \{2, \ldots, \lambda_1\}$ and $k \in \{1, \ldots, \lambda'_i\}$ such that

$$x_{ik} \notin \langle x_{11}, \ldots, x_{1\lambda'_1}\rangle.$$

Also

$$\langle x_{11}, \ldots, x_{1\lambda'_1}\rangle \nsubseteq \langle x_{i1}, \ldots, x_{i\lambda'_i}\rangle.$$

In fact if $\langle x_{11}, \ldots, x_{1\lambda'_1}\rangle \subseteq \langle x_{i1}, \ldots, x_{i\lambda'_i}\rangle$ then

$$\dim\langle x_{11}, \ldots, x_{1\lambda'_1}\rangle \leqslant \dim\langle x_{i1}, \ldots, x_{i\lambda'_i}\rangle$$

so $\lambda'_1 \leqslant \lambda'_i$ which implies $\lambda'_1 = \lambda'_i$ and so $x_{ik} \in \langle x_{11}, \ldots, x_{1\lambda'_1}\rangle$.

Contradiction.

We can conclude that there exists $j \in \left\{1, \ldots, \lambda'_1\right\}$ such that

$$x_{1j} \notin \langle x_{i1}, \ldots, x_{i\lambda'_i}\rangle.$$

So $\left(x_{11}, \ldots, x_{1j-1}, x_{ik}, x_{1j+1}, \ldots, x_{1\lambda'_1}\right)$ and $\left(x_{i1}, \ldots, x_{ik-1}, x_{1j}, x_{ik+1}, \ldots, x_{i\lambda'_i}\right)$ are linearly independent families.

Consequently

$$\left\{x_{11}, \ldots, x_{1\lambda'_1}\right\}\dot{\cup}\left\{x_{21}, \ldots, x_{2\lambda'_2}\right\}\dot{\cup}\cdots\dot{\cup}\left\{x_{\lambda_1 1}, \ldots, x_{\lambda_1\lambda'_{\lambda_1}}\right\}$$

and

$$\begin{aligned}&\left\{x_{11}, \ldots, x_{1j-1}, x_{ik}, x_{1j+1}, \ldots, x_{1\lambda'_1}\right\}\dot{\cup}\left\{x_{21}, \ldots, x_{2\lambda'_2}\right\}\\ &\quad\dot{\cup}\cdots\dot{\cup}\left\{x_{i1}, \ldots, x_{ik-1}, x_{1j}, x_{ik+1}, \ldots, x_{i\lambda'_i}\right\}\\ &\quad\dot{\cup}\cdots\dot{\cup}\left\{x_{\lambda_1 1}, \ldots, x_{\lambda_1\lambda'_{\lambda_1}}\right\}\end{aligned}$$

are two λ'-colorings of $(x_1, \ldots, x_m)$ satisfying the referred conditions. $\square$

Suppose that z is not critical. Then, by the Theorem 4.3 and $z \neq 0$, we have $\dim\langle x_1, \ldots, x_m\rangle = \dim W(z) > \lambda'_1$. According now to the proved fact and the Proposition 5.9 we conclude that

$$\mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \subseteq \langle x_{11}, \ldots, x_{1\lambda'_1}\rangle \cap \langle y_{11}, \ldots, y_{1\lambda'_1}\rangle.$$

This implies that

$$\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \leqslant \dim(\langle x_{11}, \ldots, x_{1\lambda'_1}\rangle \cap \langle y_{11}, \ldots, y_{1\lambda'_1}\rangle) < \lambda'_1,$$

a contradiction. □

Next proposition gives us a necessary and sufficient condition for the criticality of decomposable tensors.

Corollary 1. *Let* $0 \neq z = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ *be a decomposable tensor of* V_λ. *Then* z *is critical if and only if there exists a linearly independent family* $\left(v_1, \ldots, v_{\lambda'_1}\right)$ *with elements in* $\mathrm{Ann}^{(x_1,\ldots,x_m)}(z)$.

Proof. Suppose that z is critical. By Theorems 4.4 and 4.3 we conclude that

$$\mathrm{Ann}^{(x_1,\ldots,x_m)}(z) = W(z).$$

Consequently $\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) = \lambda'_1$, so there exists λ'_1 linearly independent vectors in $\mathrm{Ann}^{(x_1,\ldots,x_m)}(z)$.

Conversely, let $(v_1, \ldots, v_{\lambda'_1})$ be a family of linearly independent vectors in $\mathrm{Ann}^{(x_1,\ldots,x_m)}(z)$. Corollary 1 of Proposition 5.9 gives us

$$\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) \leqslant \lambda'_1.$$

So, we can conclude that $\dim \mathrm{Ann}^{(x_1,\ldots,x_m)}(z) = \lambda'_1$. Then, by the Theorem 5.5, we have that z is critical. □

6. λ-Plücker polynomials

The main purpose of this section is to construct a family of polynomials characterizing the criticality of a decomposable tensor of V_λ. The idea behind this construction is to use Corollary 1 to Theorem 5.5 to extend the argument referred by Marcus in [6].

We start with some basic computations. Recall we are fixing a basis $(e_1, \ldots, e_n)$ of V. Consider a tensor $z \in \otimes^m V$,

$$z = \sum_{\alpha \in \Gamma_{m,n}} a_\alpha e_\alpha^{\otimes}.$$

Let $\nu \in \Gamma_{m-1,n}$ and $t \in \{1, \ldots, m\}$, we denote by $u_{t,\nu}^{(z)}$ or briefly by $u_{t,\nu}$ the vector of V,

$$u_{t,\nu} := \sum_{j=1}^{n} a_{\nu \overset{t}{\hookleftarrow} j} e_j.$$

Let $\gamma \in \Gamma_{m+1,n}$. Let $\pi_1^{(\gamma)}, \ldots, \pi_{s_\gamma}^{(\gamma)}$ be a system of representatives of the right cosets of H_γ in S_{m+1}, i. e.,

$$S_{m+1} := H_\gamma \pi_1^{(\gamma)} \dot{\cup} \cdots \dot{\cup} H_\gamma \pi_{s_\gamma}^{(\gamma)}. \tag{12}$$

For $i \in \{1, \ldots, s_\gamma\}$ we denote the mapping $\gamma \pi_i^{(\gamma)}$ by $\gamma^{(i)}$.

It can be easily seen that $(e^{\otimes}_{\gamma^{(i)}})_{i=1,\ldots,s_\gamma}$ is a basis of the orbital subspace associated to γ, i.e.,

$$\langle e^{\otimes}_{\gamma\sigma} : \sigma \in S_{m+1} \rangle = \langle e^{\otimes}_{\gamma^{(1)}}, \ldots, e^{\otimes}_{\gamma^{(s_\gamma)}} \rangle.$$

Therefore, if $l \in \{1, \ldots, s_\gamma\}$, we have

$$\begin{aligned} T_\chi(e^{\otimes}_{\gamma^{(l)}}) &= \frac{\chi(\mathrm{id})}{(m+1)!} \sum_{k=1}^{s_\gamma} \left(\sum_{\tau \in H_\gamma} \chi(\pi_k^{-1} \tau \pi_l) \right) e^{\otimes}_{\gamma^{(k)}} \\ &= \sum_{k=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} e^{\otimes}_{\gamma^{(k)}}, \end{aligned} \tag{13}$$

where $\mathfrak{c}_{\gamma,k,l}$ denotes

$$\mathfrak{c}_{\gamma,k,l} := \frac{\chi(\mathrm{id})}{(m+1)!} \sum_{\tau \in H_\gamma} \chi(\pi_k^{-1} \tau \pi_l).$$

Definition 6.8. Let $\gamma \in \Gamma_{m+1,n}$, $\nu \in \Gamma_{m-1,n}$, $t \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, s_\gamma\}$. The polynomial of $\mathbb{C}[X_\alpha : \alpha \in \Gamma_{m,n}]$

$$f_{\gamma,\nu,t,k}(X_\alpha : \alpha \in \Gamma_{m,n}) := \sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} X_{\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1)} X_{\gamma^{(l)}_{m+1}}$$

is called the λ-*Plücker polynomial* associated with (γ, ν, t, k).

We denote by η the element of $\Gamma_{m,n}$, $\eta = (1, \ldots, m)$. If $A = (a_{ij}) \in \mathbb{C}^{m \times n}$ and $\alpha \in \Gamma_{m,n}$, we denote by $A[\eta|\alpha]$ the $m \times m$ matrix whose jth column is the column $\alpha(j)$ of A, $j = 1, \ldots, m$; i.e., the (i, j) entry of $A[\eta|\alpha]$ is $a_{i,\alpha(j)}$, $i, j = 1, \ldots, m$.

If $B = (b_{ij}) \in \mathbb{C}^{m \times m}$, we denote by $d_\lambda(B)$ the value of the generalized matrix function d_λ on B,

$$d_\lambda(B) := \sum_{\sigma \in S_m} \lambda(\sigma) \prod_{t=1}^{m} b_{t,\sigma(t)}.$$

The Hadamard function on B will be denoted by $\mathfrak{h}(B)$, i.e., $\mathfrak{h}(B) := \prod_{i=1}^{m} b_{ii}$.

Next result is technical and prepares the computations needed for the main results of this section.

Lemma 6.3. *Let z be an element of $\otimes^m V$,*

$$z = \sum_{\alpha\in\Gamma_{m,n}} a_\alpha e_\alpha^\otimes.$$

Let u be a vector of V, $u = \sum_{j=1}^n c_j e_j$. The following equality holds:

$$T_\chi(z\otimes u) = \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma} \left(\sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} c_{\gamma^{(l)}(m+1)} a_{\gamma^{(l)}_{m+1}} \right) e^\otimes_{\gamma^{(k)}}.$$

Proof. By the assumptions of the theorem we have

$$\begin{aligned} T_\chi(z\otimes u) &= T_\chi\left(\left(\sum_{\alpha\in\Gamma_{m,n}} a_\alpha e_\alpha^\otimes\right)\otimes\left(\sum_{j=1}^n c_j e_j\right)\right) \\ &= \sum_{j=1}^n \sum_{\alpha\in\Gamma_{m,n}} c_j a_\alpha T_\chi(e_\alpha^\otimes \otimes e_j). \end{aligned}$$

Therefore, since $\Gamma_{m+1,n} = \Gamma_{m,n}\times\{1,\ldots,n\}$, we get,

$$T_\chi(z\otimes u) = \sum_{\gamma\in\Gamma_{m+1,n}} c_{\gamma(m+1)} a_{\gamma_{m+1}} T_\chi(e_\gamma^\otimes).$$

As $G_{m+1,n}$ is a system of distinct representatives of the orbits for the action of S_{m+1} on $\Gamma_{m+1,n}$ and due to (12) and (13) we obtain, from the previous equalities,

$$\begin{aligned} T_\chi(z\otimes u) &= \sum_{\gamma\in G_{m+1,n}} \sum_{l=1}^{s_\gamma} c_{\gamma^{(l)}(m+1)} a_{\gamma^{(l)}_{m+1}} T_\chi(e^\otimes_{\gamma^{(l)}}) \\ &= \sum_{\gamma\in G_{m+1,n}} \sum_{l=1}^{s_\gamma} c_{\gamma^{(l)}(m+1)} a_{\gamma^{(l)}_{m+1}} \sum_{k=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} e^\otimes_{\gamma^{(k)}} \\ &= \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma} \left(\sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} c_{\gamma^{(l)}(m+1)} a_{\gamma^{(l)}_{m+1}} \right) e^\otimes_{\gamma^{(k)}}. \qquad \square \end{aligned}$$

Lemma 6.3 can be restated in view of definition of λ-Plücker polynomial as follows:

Corollary 2. *If $\nu\in\Gamma_{m-1,n}$ and $t\in\{1,\ldots,m\}$, we have the following equality*

$$T_\chi(z\otimes u_{t,\nu}) = \sum_{\gamma\in G_{m+1,n}} \sum_{k=1}^{s_\gamma} f_{\gamma,\nu,t,k}(a_\alpha : \alpha\in\Gamma_{m,n}) e^\otimes_{\gamma^{(k)}}.$$

Definition 6.9. Let γ be an element of $\Gamma_{m+1,n}$. Let ν be an element of $\Gamma_{m-1,n}$, and t and k be positive integers respectively in $\{1,\ldots,m\}$ and $\{1,\ldots,s_\gamma\}$. We denote by $F_{\gamma,\nu,t,k}$ the polynomial of $\mathbb{C}[X_\alpha : \alpha \in \Gamma_{m,n}]$,

$$F_{\gamma,\nu,t,k} := \sum_{\sigma\in S_m} \lambda(\sigma) f_{\gamma,\xi_{\nu,\sigma,t,\gamma^{(l)}(m+1)},\sigma^{-1}(t),k},$$

where $\xi_{\nu,\sigma,t,j} := [(\nu \overset{t}{\hookleftarrow} j)\sigma]_{\sigma^{-1}(t)}$.

We denote by D_α the polynomial of $\mathbb{C}[X_\alpha : \alpha \in \Gamma_{m,n}]$,

$$D_\alpha(X_\beta : \beta \in \Gamma_{m,n}) := \sum_{\sigma\in S_m} \lambda(\sigma) X_{\alpha\sigma}.$$

Proposition 6.10. *Let* $\gamma \in \Gamma_{m+1,n}$, $\nu \in \Gamma_{m-1,n}$, $t \in \{1,\ldots,m\}$ *and* $k \in \{1,\ldots,s_\gamma\}$. *Then, we have*

$$F_{\gamma,\nu,t,k} = \sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} D_{\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1)} X_{\gamma^{(l)}_{m+1}}.$$

Proof. By definitions, we have

$$\begin{aligned} F_{\gamma,\nu,t,k} &= \sum_{\sigma\in S_m} \lambda(\sigma) f_{\gamma,[(\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1))\sigma]_{\sigma^{-1}(t)},\sigma^{-1}(t),k} \\ &= \sum_{\sigma\in S_m} \lambda(\sigma) \sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} X_{[(\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1))\sigma]_{\sigma^{-1}(t)} \overset{\sigma^{-1}(t)}{\hookleftarrow} \gamma^{(l)}(m+1)} X_{\gamma^{(l)}_{m+1}} \\ &= \sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} \left(\sum_{\sigma\in S_m} \lambda(\sigma) X_{(\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1))\sigma} \right) X_{\gamma^{(l)}_{m+1}} \\ &= \sum_{l=1}^{s_\gamma} \mathfrak{c}_{\gamma,k,l} D_{\nu \overset{t}{\hookleftarrow} \gamma^{(l)}(m+1)} X_{\gamma^{(l)}_{m+1}}. \qquad \square \end{aligned}$$

Next lemma gives the connection between the polynomials D_α and the generalized matrix functions as it was done in [2].

Lemma 6.4. *Let* $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ *and*

$$x_i = \sum_{j=1}^{n} a_{ij} e_j \quad i = 1,\ldots,m.$$

Let z *be the decomposable tensor*

$$z = x_1 \otimes \cdots \otimes x_m = \sum_{\beta\in\Gamma_{m,n}} a_\beta e^{\otimes}_\beta.$$

Then the following equality holds

$$D_\alpha(a_\beta : \beta \in \Gamma_{m,n}) = d_\lambda(A[\eta|\alpha]).$$

Proof. Since $a_\beta = \mathfrak{h}(A[\eta|\beta]), \forall \beta \in \Gamma_{m,n}$, then

$$\begin{aligned} D_\alpha(a_\beta : \beta \in \Gamma_{m,n}) &= \sum_{\sigma \in S_m} \lambda(\sigma)\mathfrak{h}(A[\eta|\alpha\sigma]) \\ &= \sum_{\sigma \in S_m} \lambda(\sigma) \prod_{t=1}^{m} a_{t,\alpha\sigma(t)} \\ &= d_\lambda(A[\eta|\alpha]). \qquad \square \end{aligned}$$

A special linearly independent family of vectors is constructed in the following proposition.

Proposition 6.11. *Let* $A = (a_{ij}) \in \mathbb{C}^{m \times n}$,

$$x_i = \sum_{j=1}^{n} a_{ij} e_j \quad i = 1, \ldots, m.$$

Assume that $z^* = T_\lambda(x_1 \otimes \cdots \otimes x_m) \neq 0$. *Let* $\omega \in \operatorname{supp}(z^*)$ *such that* $M(\omega)$ *is maximal for the majorization order of* $\{M(\alpha) : \alpha \in \operatorname{supp}(z^*)\}$. *Let* $\omega(\{1, \ldots, m\}) = \{p_1, \ldots, p_l\}$, $(|\omega^{-1}(p_1)| \geqslant \cdots \geqslant |\omega^{-1}(p_l)|)$ *and* $r_i = \min \omega^{-1}(p_i)$, $i = 1, \ldots, l$. *Then*

$$v_i := u^{(z^*)}_{r_i,\omega_{r_i}} = \sum_{j=1}^{n} \frac{\lambda(\mathrm{id})}{m!} d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} j])e_j, \quad i = 1, \ldots, l$$

is a linearly independent family.

Proof. We begin by proving the following:

Fact. *If* $j < i$, *then* $M(\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j) \succeq\!\!\!\!\!/\; M(\omega)$.

Proof. If $j < i$, we have

$$\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j = (\omega(1), \ldots, \omega(r_i - 1), p_j, \omega(r_i + 1), \ldots, \omega(m)).$$

Let $t = \min\{s : |\omega^{-1}(p_s)| = |\omega^{-1}(p_j)|\}$ and $k = \max\{s : |\omega^{-1}(p_s)| = |\omega^{-1}(p_i)|\}$. Then,

$$\begin{aligned} M(\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j) = (&|\omega^{-1}(p_1)|, \ldots, |\omega^{-1}(p_{t-1})|, |\omega^{-1}(p_j)| + 1, |\omega^{-1}(p_t)| \\ &, \ldots, |\omega^{-1}(p_k)|, |\omega^{-1}(p_i)| - 1, |\omega^{-1}(p_{k+1})|, \ldots, |\omega^{-1}(p_l)|). \end{aligned}$$

Therefore,

$$M(\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j) \gneqq M(\omega). \qquad \square$$

Then, we have

$$\begin{aligned} v_i &= \sum_{j=1}^{n} \frac{\lambda(\mathrm{id})}{m!} d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} j])e_j \\ &= \frac{\lambda(\mathrm{id})}{m!}(d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} p_1])e_{p_1} + \cdots + d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} p_l])e_{p_l} \\ &\quad + \sum_{j \notin \{p_1,\ldots,p_l\}} d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} j])e_j). \end{aligned}$$

Since for $j < i$, we have $M(\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j) \gneqq M(\omega)$, we can conclude that $\omega_{r_i} \overset{r_i}{\hookleftarrow} p_j \notin \mathrm{supp}(z^*)$ if $j < i$. Then

$$\begin{aligned} v_i &= \frac{\lambda(\mathrm{id})}{m!}(d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} p_i])e_{p_i} + \cdots + d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} p_l])e_{p_l} \\ &\quad + \sum_{j \notin \{p_1,\ldots,p_l\}} d_\lambda(A[\eta|\omega_{r_i} \overset{r_i}{\hookleftarrow} j])e_j). \end{aligned}$$

But, by definition, $\omega_{r_i} \overset{r_i}{\hookleftarrow} p_i = \omega$, so we have that $(v_1, \ldots, v_l)$ is linearly independent. $\square$

Lemma 6.5. *Let ν be an element of $\Gamma_{m-1,n}$ and $t \in \{1, \ldots, m\}$.*
Let $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ and

$$x_i = \sum_{j=1}^{n} a_{ij} e_j \quad i = 1, \ldots, m.$$

Then, if

$$z = x_1 \otimes \cdots \otimes x_m = \sum_{\alpha\in\Gamma_{m,n}} a_\alpha e_\alpha^\otimes,$$

we have

$$u_{t,\nu} \in \langle x_t \rangle.$$

Proof. It is well known [5] that the coeficient of $x_1 \otimes \cdots \otimes x_m$ in $e_\alpha^\otimes$ is the value of the Hadamard function on the matrix $A[\eta|\alpha]$, i.e.,

$$x_1 \otimes \cdots \otimes x_m = \sum_{\alpha\in\Gamma_{m,n}} \mathfrak{h}(A[\eta|\alpha])e_\alpha^\otimes.$$

Therefore taking $k = a_{1,\nu(1)} \dots a_{t-1,\nu(t-1)} a_{t+1,\nu(t)} \dots a_{m,\nu(m-1)}$, we have

$$\begin{aligned}
u_{t,\nu} &= \sum_{j=1}^{n} \mathfrak{h}(A[\eta|\nu \overset{t}{\hookleftarrow} j])e_j \\
&= \sum_{j=1}^{n} \left(\prod_{r=1}^{t-1} a_{r,\nu(r)}\right) a_{tj} \left(\prod_{r=t+1}^{m} a_{r,\nu(r-1)}\right) e_j \\
&= \left(\prod_{r=1}^{t-1} a_{r,\nu(r)} \prod_{r=t+1}^{m} a_{r,\nu(r-1)}\right) \sum_{j=1}^{n} a_{tj} e_j \\
&= k \sum_{j=1}^{n} a_{tj} e_j \\
&= k x_t. \qquad \square
\end{aligned}$$

Theorem 6.6. *Let* $0 \neq z^* = T_\lambda(x_1 \otimes \cdots \otimes x_m)$ *and*

$$z = x_1 \otimes \cdots \otimes x_m = \sum_{\alpha \in \Gamma_{m,n}} a_\alpha e_\alpha^\otimes.$$

Then z^* *is critical if and only if* $(a_\alpha : \alpha \in \Gamma_{m,n})$ *is a zero of the* λ*-Plücker polynomials associated with* (γ, ν, t, k)*, when* $\gamma \in G_{m+1,n}$, $\nu \in \Gamma_{m-1,n}$, $t \in \{1, \dots, m\}$ *and* $k \in \{1, \dots, s_\gamma\}$.

Proof. *Sufficiency.*

Let $A = (a_{ij}) \in \mathbb{C}^{m \times n}$ such that

$$x_i = \sum_{j=1}^{n} a_{ij} e_j \quad i = 1, \dots, m.$$

Then

$$z^* = T_\lambda(x_1 \otimes \cdots \otimes x_m) = \sum_{\alpha \in \Omega_\lambda} \frac{\lambda(\mathrm{id})}{m!} d_\lambda(A[\eta|\alpha]) e_\alpha^\otimes \neq 0.$$

Let $\omega \in \operatorname{supp}(z^*)$ such that $M(\omega)$ is maximal for the majorization order of

$$\{M(\alpha) : \alpha \in \operatorname{supp}(z^*)\}.$$

From Proposition 3.3, we conclude that $|\{\omega(1), \dots, \omega(m)\}| \geqslant \lambda_1'$.

Let

$$u_{t,\nu}^{(z^*)} = \sum_{j=1}^{n} \frac{\lambda(\mathrm{id})}{m!} d_\lambda(A[\eta|\nu \overset{t}{\hookleftarrow} j]) e_j.$$

According to Lemma 6.3, Lemma 6.4 and Proposition 6.10 we have

$$
\begin{aligned}
&T_\chi(z \otimes u_{t,\nu}^{(z^*)})\\
&= \sum_{\gamma \in G_{m+1,n}} \sum_{k=1}^{s_\gamma} \sum_{l=1}^{s_\gamma} \frac{\lambda(\mathrm{id})}{m!} \mathfrak{c}_{\gamma,k,l} d_\lambda(A[\eta|\nu \overset{t}{\longleftrightarrow} \gamma^{(l)}(m+1)]) a_{\gamma^{(l)}_{m+1}} e^{\otimes}_{\gamma^{(k)}}\\
&= \sum_{\gamma \in G_{m+1,n}} \sum_{k=1}^{s_\gamma} \sum_{l=1}^{s_\gamma} \frac{\lambda(\mathrm{id})}{m!} \mathfrak{c}_{\gamma,k,l} D_{\nu \overset{t}{\longleftrightarrow} \gamma^{(l)}(m+1)}(a_\alpha : \alpha \in \Gamma_{m,n}) a_{\gamma^{(l)}_{m+1}} e^{\otimes}_{\gamma^{(k)}}\\
&= \frac{\lambda(\mathrm{id})}{m!} \sum_{\gamma \in G_{m+1,n}} \sum_{k=1}^{s_\gamma} F_{\gamma,\nu,t,k}(a_\alpha : \alpha \in \Gamma_{m,n}) e^{\otimes}_{\gamma^{(k)}}\\
&= \frac{\lambda(\mathrm{id})}{m!} \sum_{\gamma \in G_{m+1,n}} \sum_{k=1}^{s_\gamma} \left(\sum_{\sigma \in S_m} \lambda(\sigma) f_{\gamma,\xi_{\nu,\sigma,t,\gamma^{(l)}(m+1)},\sigma^{-1}(t),k}(a_\alpha : \alpha \in \Gamma_{m,n}) \right) e^{\otimes}_{\gamma^{(k)}}\\
&= 0. \qquad (14)
\end{aligned}
$$

But, by Proposition 6.11, $(v_1, \ldots, v_{\lambda'_1})$ is a linearly independent family of vectors, and by (14) the vectors belongs to

$$\mathrm{Ann}^{(x_1,\ldots,x_m)}(z^*).$$

Then, by Corollary 1 to Theorem 5.5, z^* is critical.

Conversely, assume that z^* is critical.

Let $t \in \{1, \ldots, m\}$, $\nu \in \Gamma_{m-1,n}$. Then, according to Corollary 2 to Lemma 6.3, we have

$$T_\chi(z \otimes u_{t,\nu}) = \sum_{\gamma \in G_{m+1,n}} \sum_{k=1}^{s_\gamma} f_{\gamma,\nu,t,k}(a_\alpha : \alpha \in \Gamma_{m,n}) e^{\otimes}_{\gamma^{(k)}}.$$

But, by Lemma 6.5 and Theorem 4.3, for all t and all ν, $u_{t,\nu} \in \langle x_t \rangle \subseteq W(z^*)$. Since z^* is critical, it follows from Theorem 4.4, that for all t and all ν,

$$u_{t,\nu} \in \mathrm{Ann}^{(x_1,\ldots,x_m)}(z^*).$$

Then,

$$T_\chi(z \otimes u_{t,\nu}) = 0.$$

So, we have that $(a_\alpha : \alpha \in \Gamma_{m,n})$ is a root of $f_{\gamma,\nu,t,k}$, for all $\gamma \in G_{m+1,n}$, $\nu \in \Gamma_{m-1,n}$, $t \in \{1, \ldots, m\}$, $k \in \{1, \ldots, s_\gamma\}$. $\square$

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